ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 227–255

C-SEMIGROUPS AND STRONGLY CONTINUOUS SEMIGROUPS

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ABSTRACT

We show that, when A generates a C-semigroup, then there exists Y such that

$$[Im(C)] \hookrightarrow Y \hookrightarrow X,$$

and $A|_Y$, the restriction of A to Y, generates a strongly continuous semigroup, where $\hookrightarrow$ means "is continuously embedded in" and $\|x\|_{[Im(C)]} \equiv \|C^{-1}x\|$. There also exists W such that

$$[C(W)] \hookrightarrow X \hookrightarrow W,$$

and an operator B such that $A = B|_X$ and B generates a strongly continuous semigroup on W. If the C-semigroup is exponentially bounded, then Y and W may be chosen to be Banach spaces; in general, Y and W are Frechet spaces. If $\rho(A)$ is nonempty, the converse is also true.

We construct fractional powers of generators of bounded C-semigroups.

I. Introduction

Motivated by the abstract Cauchy problem,

$$\frac{d}{dt}u(t,x) = A(u(t,x)) \quad (t \geq 0), \quad u(0,x) = x, \tag{1.1}$$

* We would like to thank R. Bürger for sending preprints, and the referee for pointing out reference [37]. This research was supported by an Ohio University Research Grant.
Received April 30, 1991 and in revised form September 12, 1991

a generalization of strongly continuous semigroups, C-semigroups, has recently received much attention (see [9], [12], [14], [15], [16], [17], [18], [20], [21], [23], [32], [33], [35], [36], [37], [41], [43], [44], [45], [46], and [47]). Generating a C-semigroup corresponds to (1.1) having a unique solution, whenever $x = Cy$, for some y in the domain of A.

The class of operators that generate C-semigroups is much larger than the class of operators that generate strongly continuous semigroups. When C is chosen to be $(\lambda - A)^{-n}$, for some $n \in \mathbf{N}$, then $i\Delta$, on $L^p(\mathbf{R}^k)$, for $1 \leq p \leq \infty, p \neq 2$, may be shown to generate a C-semigroup, but not a strongly continuous semigroup (see [24] and [15]). This yields solutions of the Schrödinger equation, for all initial data in the domain of Δ^{n+1}. Much "worse" operators, corresponding to what are traditionally referred to as ill posed or improperly posed problems, generate C-semigroups. For example, if $A \equiv -\Delta$, so that (1.1) becomes the backwards heat equation, then A generates a C-semigroup. If

$$A \equiv \begin{bmatrix} 0 & I \\ -\Delta & 0 \end{bmatrix},$$

so that (1.1) becomes the Cauchy problem for the Laplace equation, then A generates a C-semigroup (see [16]).

When $Im(C)$ is dense, as is the case in the examples above, then (1.1) has a unique solution for all x in a dense set. Thus, C-semigroups have been used extensively to produce unique solutions for all initial data in a dense set. The question of whether the solutions were well-posed, in some sense, remained. The usual definition of (1.1) being well-posed is when A generates a strongly continuous semigroup (see [22], chapter 2.1).

In this paper, we show that, when A generates a C-semigroup, then there exists an "interpolation space" Y such that

$$[Im(C)] \hookrightarrow Y \hookrightarrow X,$$

and $A|_Y$, the restriction of A to Y, generates a strongly continuous semigroup. If the C-semigroup is exponentially bounded, then Y may be chosen to be a Banach space; in general, Y is a Frechet space. If $\rho(A)$ is nonempty, the converse is also true. The exponentially bounded case of this result may be found in [37].

We show that the interpolating space Y contains all initial data for which (1.1) has a solution.

We also show that there exists an "extrapolation space" W such that

$$[C(W)] \hookrightarrow X \hookrightarrow W,$$

and an operator B such that $A = B|_X$ and B generates a strongly continuous semigroup on W.

The norm on the interpolation space Y is $||x||_Y \equiv \sup\{||C^{-1}W(t)x|| \mid t \geq 0\}$, while the norm on the extrapolation space W is $||x||_W \equiv \sup\{||W(t)x|| \mid t \geq 0\}$.

This paper shows that, in a technical sense, at least if one is willing to make renormings, the concepts of C-semigroup and strongly continuous semigroup are the same. But there are important practical differences. A C-semigroup is often very easy to produce and construct, and one does not have to leave the original norm, which may be very simple or physically meaningful. The interpolation space, on which the restriction of A generates a strongly continuous semigroup, may be very difficult to construct, with a norm that is unpleasant or impossible to work with. In [13], [14], [15], [16], [17], [18], [21] and [41], there are numerous examples of C-semigroups that are constructed in the most simple-minded and intuitive manners.

Even if one's only goal is to find subspaces, Y, on which the restriction of $A, A|_Y$, generates a strongly continuous semigroup, we submit the following algorithm. First find a C-semigroup generated by A. Then use the construction of this paper (Theorems 3.1 and 3.8) to produce Y.

This paper makes it clear how C-semigroups may be used, first, to characterize all initial data that yield a solution of (1.1) (see Proposition 2.6), then, in the construction of the interpolating space Y, to find a norm with respect to which those solutions are well-posed.

This paper also shows the relationship between general C-semigroups and exponentially bounded C-semigroups. It is the same as the relationship between Frechet spaces and Banach spaces, that is, A generating a C-semigroup corresponds to a restriction of A to a Frechet space generating a strongly continuous semigroup, while generating an exponentially bounded C-semigroup corresponds to a restriction to a Banach space generating a strongly continuous semigroup. More specifically, generating a C-semigroup corresponds to an interpolating space Y with the topology of uniform convergence on compact subsets of $[0, \infty)$, while generating a bounded strongly uniformly continuous C-semigroup corresponds to Y having the topology of uniform convergence on $[0, \infty)$ (see the construction of Y in Theorems 3.1 and 3.8).

We present preliminary material on C-semigroups in section II. Our main theorems are in section III. We give examples in section IV. In section V, we give an example of how section III may be used to translate strongly continuous semigroup results to analogous results for C-semigroups, by constructing fractional powers of generators of bounded, strongly uniformly continuous C-semigroups. Section VI considers holomorphic C-semigroups and section VII uses section III to obtain perturbation results for C-semigroups (see also [47]).

All operators are linear. We will write $D(A)$ for the domain of A, $\rho(A)$ for the resolvent set. X will always be a Banach space. Y, in general, will be a Frechet space. C will always be a bounded, injective operator. If A is an operator on X and $Y \subseteq X$, then $A|_Y$ is the part of A in Y, that is,

$$D(A|_Y) \equiv \{x \in Y \cap D(A) | \, Ax \in Y\},$$

with $A|_Y x = Ax$. We will write $\{e^{tA}\}_{t\geq 0}$ to mean a strongly continuous semigroup generated by A. When W is a Banach space, we will denote by $[C(W)]$ the Banach space with norm $||x||_{[C(W)]} \equiv ||C^{-1}x||_W$; when W is a Frechet space whose topology is generated by the seminorms $\{|| \, ||_i\}_{i=1}^{\infty}$, then $[C(W)]$ will be a Frechet space whose topology is generated by the seminorms $||x||_{i,C} \equiv ||C^{-1}x||_i$ $(1 \leq i < \infty)$. $C(X)$ will also be denoted by $Im(C)$. We will denote by $[D(A)]$ the Banach space with the graph norm $||x||_{[D(A)]} \equiv ||x|| + ||Ax||$. $L(X, Y)$ will be the space of continuous, linear operators from X into Y; $L(Y)$ will mean $L(Y, Y)$.

By a **solution** of (1.1) we mean $u \in C([0,\infty), [D(A)]) \cap C^1([0,\infty), X)$, satisfying (1.1). By a mild solution we mean $u \in C([0,\infty), X)$ such that $\int_0^t u(s)\, ds \in D(A), \forall t \geq 0$, satisfying

$$u(t) = A\left(\int_0^t u(s)\, ds\right) + x,$$

$\forall t \geq 0$. We will write $Y \hookrightarrow X$ to mean that Y is continuously embedded in X, that is, $Y \subseteq X$ and the identity map from Y to X is continuous.

II. Preliminaries

Definition 2.1: The strongly continuous family of bounded operators $\{W(t)\}_{t\geq 0}$, on a Banach space X, is a **C-semigroup** if $W(0) = C$ and $W(t)W(s) = CW(t+s)$, $\forall s, t \geq 0$. ∎

The **generator** of $\{W(t)\}_{t\geq 0}$ is defined by

$$Ax = C^{-1}\left[\lim_{t\to 0}\frac{1}{t}\left(W(t)x - Cx\right)\right],$$

$$D(A) \equiv \{x \in X \,|\, \text{limit exists and is in } \mathrm{Im}(C)\}.$$

Note that when $C = I$, this is the definition of a strongly continuous semigroup generated by A. Intuitively, $W(t) = Ce^{tA} = e^{tA}C$. This intuition is realized precisely in section III.

Definition 2.2.: The strongly continuous family of bounded operators $\{T(t)\}_{t\geq 0}$, on a locally convex space Y, is a **locally equicontinuous semigroup** if $T(0) = I$, $T(t)T(s) = T(s+t), \forall s,t \geq 0$ and $\forall s < \infty, \{T(t) \,|\, 0 \leq t \leq s\}$ is equicontinuous. ∎

The generator is defined exactly as in Definition 2.1, with $C = I$.

Proposition 2.3 (see [14]): *Suppose A generates a C-semigroup $\{W(t)\}_{t\geq 0}$. Then*

(a) *A is closed.*

(b) $W(t)A \subseteq AW(t), \forall t \geq 0.$

(c) *$\frac{d}{dt}W(t)x$ exists and equals $W(t)Ax, \forall t \geq 0, x \in D(A)$.*

(d) *$\int_0^t W(s)x\,ds \in D(A)$, with $A\left(\int_0^t W(s)x\,ds\right) = W(t)x - Cx, \forall t \geq 0, x \in X$.*

(e) *(1.1) has a unique solution, $\forall x \in C\left(D(A)\right)$, given by $u(t,x) = W(t)C^{-1}x$.*

(f) *(1.1) has a unique mild solution, $\forall x \in Im(C)$, given by $u(t,x) = W(t)C^{-1}x$.*

Even when $\{W(t)\}_{t\geq 0}$ is exponentially bounded, $\rho(A)$ may be empty.

When $\{W(t)\}_{t\geq 0}$ is exponentially bounded, the following is in [32], where the operator, G, is introduced.

PROPOSITION 2.4: *Suppose $\{W(t)\}_{t\geq 0}$ is a C-semigroup generated by an extension of A, $CA \subseteq AC$ and $\mathcal{D}_W \equiv \{x \in Im(C) \,|\, t \mapsto C^{-1}W(t)x$ is differentiable at $t = 0\} \subseteq D(A)$. Then $C^{-1}AC$ generates $\{W(t)\}_{t\geq 0}$.*

PROPOSITION 2.5 (see [16], Proposition 2.9): *Suppose an extension of A generates a C-semigroup and $\rho(A)$ is nonempty. Then A is the generator.*

PROPOSITION 2.6: *Suppose A generates a C-semigroup $\{W(t)\}_{t\geq 0}$. Then (1.1) has a solution if and only if $t \mapsto C^{-1}W(t)x \in C([0,\infty),[D(A)]) \cap C^1([0,\infty),X)$. The solution is then $u(t,x) = C^{-1}W(t)x$.*

PROPOSITION 2.7: *Suppose A generates a C-semigroup $\{W(t)\}_{t\geq 0}$. Then (1.1) has a mild solution if and only if $t \mapsto C^{-1}W(t)x \in C([0,\infty),X)$ and $\int_0^t C^{-1}W(s)x\,ds \in D(A), \forall t \geq 0$. The solution is then $u(t,x) = C^{-1}W(t)x$.*

Note that (1.1) has a mild solution whenever $x \in Im(C)$. The "smoothed" solutions $C(u(t,x))$ are then well-posed, in the sense that $C(u(t,x_n))$ converges to zero, uniformly on compact sets, whenever $x_n \to 0$. In section III, we will see that, on a subspace, Y, that contains $Im(C)$, the solutions themselves are well-posed, that is, $A|_Y$ generates a strongly continuously semigroup, so that $u(t,x_n)$ converges to zero in Y, uniformly on compact sets, whenever $x_n \to 0$, in Y.

Proof of Proposition 2.4: Let $\tilde{A}$ be the generator of $\{W(t)\}_{t\geq 0}$ and let $G \equiv \tilde{A}$, restricted to $\mathcal{D}_W$. Then it is straightforward to see that $C^{-1}GC = \tilde{A}$. Thus $\tilde{A} = C^{-1}GC \subseteq C^{-1}AC$.

For the opposite inclusion, another straightforward calculation shows that $C^{-1}\tilde{A}C$ is the generator of the C^2-semigroup $\{CW(t)\}_{t\geq 0}$. The following argument shows that $\tilde{A} = C^{-1}\tilde{A}C$.

It is clear that $\tilde{A} \subseteq C^{-1}\tilde{A}C$. Suppose $x \in D(C^{-1}\tilde{A}C)$. Then, $\forall t \geq 0$, by Proposition 2.3,

$$C(W(t)x - Cx) = \int_0^t CW(s)C^{-1}\tilde{A}Cx\,ds = C\left(\int_0^t W(s)C^{-1}\tilde{A}Cx\,ds\right);$$

since C is injective, this implies that

$$(W(t)x - Cx) = \int_0^t W(s)C^{-1}\tilde{A}Cx\,ds,$$

which implies that $x \in D(\tilde{A})$, with $\tilde{A}x = C^{-1}\tilde{A}Cx$, as desired.

Thus $C^{-1}AC \subseteq C^{-1}\tilde{A}C = \tilde{A}$, so that $\tilde{A} = C^{-1}AC$. ∎

Proof of Proposition 2.6: When

$$v(t) \equiv C^{-1}W(t)x \in C([0,\infty),[D(A)]) \bigcap C^1([0,\infty),X),$$

then by Proposition 2.3(b) and (c),

$$C\frac{d}{dt}v(t) = \frac{d}{dt}Cv(t) = ACv(t) = CAv(t), \quad \forall t \geq 0,$$

so that, since C is injective, (1.1) has the solution $u(t,x) = C^{-1}W(t)x$.

Conversely, if (1.1) has a solution v, then, by Proposition 2.3(e), since $Cv(0) \in C(D(A))$, $Cv(t) = W(t)C^{-1}Cv(0) = W(t)x$. ∎

Proof of Proposition 2.7: When $v(t) \equiv C^{-1}W(t)x$ has the desired properties, then by Proposition 2.3(b) and (d),

$$CA\left(\int_0^t v(s)\,ds\right) = A\left(\int_0^t Cv(s)\,ds\right) = C(v(t)) - Cx,$$

so that, since C is injective, v is the desired solution. The converse is exactly as in the previous proof, using Proposition 2.3(f). ∎

III. Main Theorems

Our main results are in Theorems 3.16 and 3.17. The construction of the interpolating space Y, on which $A|_Y$ generates a strongly continuous semigroup, is in Theorem 3.8. In all our results, $D(A)$ may not be dense and $\rho(A)$ may be empty.

We will say that a C-semigroup $\{W(t)\}_{t\geq 0}$ is **strongly uniformly continuous** if, for all $x \in X$, the map $t \mapsto W(t)x$, from $[0,\infty)$ into X, is uniformly continuous (see Remark 3.6).

It is clear how Theorems 3.1 and 3.16 may be extended to exponentially bounded C-semigroups.

Assertions (1), (2) and (5) of the following proposition are essentially in [37], Theorem 1, with the same construction. Our result is somewhat sharper, in that the rate of growth of the C- semigroup matches the rate of growth of the strongly continuous semigroup.

PROPOSITION 3.1: *Suppose A generates a strongly uniformly continuous bounded C-semigroup, $\{W(t)\}_{t\geq 0}$. Then $\exists$ a Banach space Y such that*

(1) *$A|_Y$ generates a strongly continuous contraction semigroup,*

(2) $[C(X)] \hookrightarrow Y \hookrightarrow X$,

(3) *Y contains all initial data for which (1.1) has a bounded uniformly continuous mild solution, with*

(4) *$u(t,x) = e^{tA|_Y}x$ and*

(5) $W(t) = e^{tA|_Y}C, \forall t \geq 0$.

Y may be chosen to be

$$\{x \in X \,|\, t \mapsto C^{-1}W(t)x \text{ is a uniformly continuous map from } [0,\infty) \text{ into } X \text{ and } \|x\|_Y < \infty\},$$

where $\|x\|_Y \equiv \sup\{\|C^{-1}W(t)x\| \,|\, t \geq 0\}$.

COROLLARY 3.2: *Suppose A generates a strongly uniformly continuous bounded C-semigroup $\{W(t)\}_{t\geq 0}$. Then $\exists$ a Banach space W and an operator B such that*

(1) *C extends to a bounded operator, $\tilde{C}$, on W,*
(2) *B generates a strongly continuous contraction semigroup on W, such that $e^{tB}\tilde{C} = \tilde{C}e^{tB}, \forall t \geq 0$,*
(3) *$Ce^{tB}x = W(t)x, \forall t \geq 0, x \in X$,*
(4) *$A = B|_X$ and*
(5) *$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W$.*

W may be chosen so that

$$\|x\|_W = \sup\{\|W(t)x\| \,|\, t \geq 0\}, \qquad \forall x \in X.$$

Remark 3.3: Theorem 0.2, in [2], has a result similar to Corollary 3.2, for exponentially bounded n-times integrated semigroups of type $w > 0$ (B generates an exponentially bounded n-times integrated semigroup of type $w > 0$ if and only if $(w,\infty) \subseteq \rho(B)$ and B generates an exponentially bounded $(\lambda - B)^{-n}$-semigroup, $\forall \lambda > w$; see [15]). By choosing $C \equiv (\lambda - A)^{-n}$, Theorem 0.2, in [2], is a corollary of Corollary 3.2. ∎

Another construction of interpolation spaces, for a class of operators related to integrated semigroups, appears in [40].

Remark 3.4: In [27], an interpolating space, Z, defined to be the **Hille–Yosida space**, is constructed, for an arbitrary operator, A, so that $A|_Z$ generates a strongly continuous contraction semigroup. When A generates a bounded C-semigroup, then it may be shown that $C(D(A)) \subseteq Z$; however, even when $D(A)$ is dense, it is not clear if $Im(C) \subseteq Z$. ∎

Remark 3.5: In [12], section 6, it is stated that the construction in [34] may be used to construct an interpolating space Y satisfying (1) and (2) of Theorem 3.1, when $Im(C)$ is dense and A generates an exponentially bounded C-semigroup. However, the construction there required that $(\lambda - A)^{-1}$ leave $Im(C)$ invariant; the following example shows that this is not true, in general. Let

$$A \equiv \begin{bmatrix} 0 & C^{-1} \\ 0 & 0 \end{bmatrix}, \qquad D(A) \equiv X \times Im(C).$$

Then A generates the exponentially bounded C-semigroup

$$W(t) \equiv \begin{bmatrix} C & t \\ 0 & C \end{bmatrix},$$

and

$$(\lambda - A)^{-1} = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda^2} C^{-1} \\ 0 & \frac{1}{\lambda} \end{bmatrix},$$

with the same domain as A, $\forall \lambda \neq 0$, which clearly does not leave $Im(C) \times Im(C)$ invariant. ∎

Remark 3.6: If A, in Theorem 3.1, is densely defined and $\{W(t)\}_{t \geq 0}$ is bounded, then $\{W(t)\}_{t \geq 0}$ is strongly uniformly continuous. The same is true if $\{W(t)\}_{t \geq 0}$ is stable. It is not clear if all bounded C-semigroups are strongly uniformly continuous. If $\exists Y$ as in (1) and (2) of Theorem 3.1, then $\{W(t)\}_{t \geq 0}$ is strongly uniformly continuous (see Proposition 3.11). ∎

Remark 3.7: Note that, in Theorem 3.1, Y does not contain all initial data for which (1.1) has a solution, in fact, the following example shows that, in general, there will exist no Banach space satisfying (2), (3) and (4), with $A|_Y$ generating a strongly continuous semigroup, that will contain all such initial data.

Let

$$X \equiv C_0(\mathbf{R}) \cap C_0^1([0, \infty)), \quad A \equiv -\frac{d}{dx}, \quad D(A) \equiv \{f \in X \,|\, f' \in X\}.$$

Then A generates a bounded strongly uniformly continuous $(1-A)^{-1}$-semigroup, defined by

$$W(t)f(x) \equiv \left((1 - A)^{-1} f\right)(x - t).$$

If Y were as in Theorem 3.1 and u were defined by (4), then u would be exponentially bounded. However, there are solutions of (1.1) that are not exponentially bounded. Choose $f \in D(A)$ such that, $\forall k \in \mathbf{N}$,

$$\sup\{|f'(-x)| \,|\, k - 1 \leq x \leq k\} \geq e^{k^2}.$$

Then

$$u(t, f)(x) \equiv f(x - t)$$

is a solution of (1.1) and

$$||u(t, f)|| \geq e^{t^2}, \qquad \forall t \geq 0.$$

Hence, to obtain an interpolating space containing all initial data for which (1.1) has a mild solution, we must let Y be a Frechet space. ∎

THEOREM 3.8: *Suppose A generates a C-semigroup $\{W(t)\}_{t\geq 0}$. Then $\exists$ a Frechet space Y such that*

(1) *$A|_Y$ generates a locally equicontinuous semigroup,*

(2) *$[C(X)] \hookrightarrow Y \hookrightarrow X$,*

(3) *Y contains all initial data for which (1.1) has a mild solution, with*

(4) *$u(t, x) = e^{tA|_Y} x$ and*

(5) *$W(t) = e^{tA|_Y} C, \forall t \geq 0$.*

Y may be chosen to be

$$\{x \in X \,|\, t \mapsto C^{-1}W(t)x \text{ is a continuous map from } [0, \infty) \text{ into } X\},$$

with the topology of Y generated by the seminorms

$$||x||_{i,j} \equiv \sup\{||C^{-1}W(t)x|| \,|\, t \in [r_i, r_j]\},$$

where $\{r_i\}_{i=1}^{\infty}$ is a denumeration of the nonnegative rational numbers.

COROLLARY 3.9: *Suppose A generates a C-semigroup $\{W(t)\}_{t\geq 0}$. Then $\exists$ a Frechet space W and an operator B such that*

(1) *C extends to a bounded operator, $\tilde{C}$, on W,*

(2) *B generates a locally equicontinuous semigroup on W, such that $e^{tB}\tilde{C} = \tilde{C}e^{tB}, \forall t \geq 0$,*

(3) *$\tilde{C}e^{tB}x = W(t)x, \forall t \geq 0, x \in X$,*

(4) *$A = B|_X$ and*

(5) *$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W$.*

W may be chosen to have topology generated by the seminorms

$$||x||_{i,j} = \sup\{||W(t)x|| \mid t \in [r_i, r_j]\}, \qquad \forall x \in X,$$

where $\{r_i\}_{i=1}^{\infty}$ is a denumeration of the nonnegative rational numbers.

Remark 3.10: The construction of Theorem 3.8 could also be done when A generates a semigroup of unbounded operators, $\{T(t)\}_{t\geq 0}$, as in [26]. If

$$\mathcal{D} \equiv \left\{ x \in \bigcap_{s,t>0} D(T(t)T(s)) \mid t \mapsto T(t)x \text{ is continuous,} \right.$$
$$\left. T(t)T(s)x = T(s+t)x, \forall s,t > 0 \right\},$$

as in [26], then one would obtain a Frechet space, Y, containing $\mathcal{D}$, such that $\{T(t)|_Y\}_{t\geq 0}$ is a locally equicontinuous semigroup. Since $\mathcal{D}$ does not have a topology, one would have less information about the topology of Y, than is contained in Theorem 3.8. Also, it is not clear what the relationship between the generator of $\{T(t)|_Y\}_{t\geq 0}$ and $A|_Y$ is, where A is the generator of $\{T(t)\}_{t\geq 0}$, as defined in [26]. ∎

In Propositions 3.11, 3.12, 3.14 and 3.15, note that, when $\rho(A)$ is nonempty, then A itself is the generator, by Proposition 2.5.

PROPOSITION 3.11: *Suppose $CA \subseteq AC$ and $\exists$ a Banach space Y such that*

$$[C(X)] \hookrightarrow Y \hookrightarrow X$$

and $A|_Y$ generates a strongly continuous contraction semigroup. Then an extension of A generates a bounded strongly uniformly continuous C-semigroup on X.

PROPOSITION 3.12: *Suppose $\exists$ a Banach space W, a bounded extension, $\tilde{C}$, of C, on W and an operator B such that*

$$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W,$$

B generates a strongly continuous contraction semigroup on W, $e^{tB}\tilde{C} = \tilde{C}e^{tB}$, $\forall t \geq 0$ and $A = B|_X$. Then an extension of A, $C^{-1}AC$, generates a bounded strongly uniformly continuous C-semigroup on X.

Remark 3.13: By choosing $C \equiv (\lambda - B)^{-k}$, and using Proposition 2.5 and the equivalence between generating a k-times integrated semigroup and generating a $(\lambda - B)^{-k}$-semigroup (see Remark 3.3), Theorem 0.1, in [2], is a Corollary of Proposition 3.12. ∎

PROPOSITION 3.14: *Suppose $CA \subseteq AC$ and $\exists$ a Frechet space Y such that*

$$[C(X)] \hookrightarrow Y \hookrightarrow X$$

and $A|_Y$ generates a locally equicontinuous semigroup. Then an extension of A generates a C-semigroup on X.

PROPOSITION 3.15: *Suppose $\exists$ a Frechet space W, a bounded extension, $\tilde{C}$, of C, to W and an operator B such that*

$$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W,$$

B generates a locally equicontinuous semigroup on W, $e^{tB}\tilde{C} = \tilde{C}e^{tB}$ and $A = B|_X$. Then an extension of A, $C^{-1}AC$, generates a C-semigroup on X.

The equivalence of (a) and (b) in the following theorem is in [37], Theorem 1.

THEOREM 3.16: *The following are equivalent.*

(a) *A generates a bounded strongly uniformly continuous C-semigroup.*

(b) *$A = C^{-1}AC$ and $\exists$ a Banach space Y such that*

$$[C(X)] \hookrightarrow Y \hookrightarrow X,$$

and $A|_Y$ generates a strongly continuous contraction semigroup.

(c) *$\exists$ a Banach space W and an operator B such that B generates a strongly continuous contraction semigroup on W, C extends to a bounded operator, $\tilde{C}$, on W, $e^{tB}\tilde{C} = \tilde{C}e^{tB}, \forall t \geq 0$,*

$$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W,$$

and $A = B|_X$.

THEOREM 3.17: *The following are equivalent.*

(a) *A generates a C-semigroup.*

(b) $A = C^{-1}AC$ *and* $\exists$ *a Frechet space* Y *such that*

$$[C(X)] \hookrightarrow Y \hookrightarrow X,$$

and $A|_Y$ *generates a locally equicontinuous semigroup.*

(c) $\exists$ *a Frechet space* W *and an operator* B *such that* B *generates a locally equicontinuous semigroup on* W, C *extends to a bounded operator* $\tilde{C}$, *on* W, $e^{tB}\tilde{C} = \tilde{C}e^{tB}, \forall t \geq 0$,

$$[\tilde{C}(W)] \hookrightarrow X \hookrightarrow W,$$

and $A = B|_X$.

The following improves Corollary 0.3, in [2], by removing the hypothesis that $D(A)$ be dense. The equivalence of (a) and (b) is in [37], Corollary 2.

COROLLARY 3.18: *Suppose* $\rho(A)$ *is nonempty. Then the following are equivalent.*

(a) *A generates an exponentially bounded n-times integrated semigroup.*

(b) $\exists$ *a Banach space* Y *such that* $A|_Y$ *generates a strongly continuous semigroup and*

$$[D(A^n)] \hookrightarrow Y \hookrightarrow X.$$

(c) $\exists$ *a Banach space* W *and an operator* B *such that* B *generates a strongly continuous semigroup on* W,

$$[D(B^n)] \hookrightarrow X \hookrightarrow W,$$

and $A = B|_X$.

For Lemma 3.19 and Corollary 3.20, assume $\{W(t)\}_{t\geq 0}$ is a C-semigroup.

LEMMA 3.19: *If* $W(t)x \in Im(C)$, *then* $W(s)C^{-1}W(t)x = W(s+t)x$, $\forall s \geq 0$.

Proof: $C\left[W(s)C^{-1}W(t)x\right] = W(s)CC^{-1}W(t)x = W(s)W(t)x = CW(s+t)x$, thus this follows from the fact that C is injective.

COROLLARY 3.20: *If $W(t)x \in Im(C), \forall t \geq 0$, then $W(s)C^{-1}W(t)x \in Im(C)$, $\forall s,t \geq 0$.*

Proof of Proposition 3.1: Assertions (1) and (2) follow as in the proof of Theorem 1, in [37], with $e^{tA|_Y} \equiv C^{-1}W(t)$. (3) and (4) follow from Proposition 2.7. (5) follows from the fact that C is injective and $CW(t) = W(t)C = Ce^{tA|_Y}C$. ∎

Proof of Corollary 3.2: Let Z be the completion of X, with respect to the norm $||x||_Z \equiv ||Cx||$. Extend C to Z by defining $\tilde{C}z = \lim_{n\to\infty} Cx_n$, with the limit taken in X, whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X converging to z, in Z. It is not hard to see that $\tilde{C}$ is bounded and injective on Z, and $\tilde{C}(Z)$ equals the closure, in X, of $C(X)$. For any $t \geq 0$, extend $W(t)$ to Z by $W(t)z \equiv \tilde{C}^{-1}W(t)Cz$; note that, since $W(t)$ is bounded and commutes with C, $W(t)Cz \in \tilde{C}(Z), \forall z \in Z$.

W is now constructed from Z exactly as Y was constructed from X, in Theorem 3.1. Note that

$$||x||_W \equiv \sup_{t\geq 0} ||\tilde{C}^{-1}W(t)x||_Z = \sup_{t\geq 0} ||W(t)x||.$$

Thus $X \subseteq W$ and (5) is clear.

$B \equiv A_1|_W$, where A_1 is the generator of the extension of $\{W(t)\}_{t\geq 0}$ to Z; by Theorem 3.1, B generates a strongly continuous contraction semigroup on W. To see that $A = B|_X$, suppose $x \in D(B|_X)$. Then, $\forall t > 0$,

$$\frac{1}{t}\left(W(t)x - Cx\right) = \tilde{C}\left(\frac{1}{t}(e^{tB}x - x)\right),$$

which converges to CBx, as $t \to 0$, in X, because $[\tilde{C}(W)] \hookrightarrow X$. Thus $x \in D(A)$ and $Ax = Bx$, so that $B|_X \subseteq A$. Conversely, suppose $x \in D(A)$. Then, $\forall t \geq 0$,

$$\tilde{C}e^{tB}x - \tilde{C}x = W(t)x - \tilde{C}x = \int_0^t W(s)Ax\,ds = \tilde{C}\left(\int_0^t e^{sB}Ax\,ds\right),$$

so that, since $\tilde{C}$ is injective,

$$e^{tB}x - x = \int_0^t e^{sB}Ax\,ds,$$

which implies that $x \in D(B)$ and $Bx = Ax \in X$, so that $x \in D(B|_X)$, as desired. ∎

Proof of Theorem 3.8: Y is a locally convex space generated by a countable family of seminorms, thus to show that Y is a Frechet space, it is sufficient to show that Y is complete.

Suppose $\{y_n\}$ is a Cauchy sequence in Y. Since $W(0) = C$, it is clear that $\exists x \in X$ such that $y_n \to x$, in X. Fix i and j. For $t \in [r_i, r_j]$, $\exists z(t) \in X$ such that $C^{-1}W(t)y_n$ converges to $z(t)$, and $W(t)y_n$ converges to $W(t)x$, in X, both uniformly on $[r_i, r_j]$. Since C is bounded, $Cz(t) = W(t)x$, so that $W(t)x \in Im(C), \forall t \in [r_i, r_j]$. The continuity of $t \mapsto C^{-1}W(t)x = z(t)$, and the fact that $\|y_n - x\|_{i,j}$ converges to 0, both follow from the uniform convergence of $C^{-1}W(t)y_n$. Since i and j were arbitrary, $x \in Y$ and y_n converges to x, in Y.

By Lemma 3.19 and Corollary 3.20,

$$T(s) \equiv C^{-1}W(s)$$

maps Y into itself, $\forall s \geq 0$. By Lemma 3.19, for $s \geq 0, i, j \in \mathbf{N}, x \in Y$,

$$\|T(s)x\|_{i,j} = \sup\{\|C^{-1}W(t)x\| \mid t \in [r_i + s, r_j + s]\} \leq \|x\|_{k,l},$$

when $r_k < r_i + s$ and $r_l > r_j + s$. Thus $T(s) \in L(Y)$. The strong continuity of $\{T(s)\}_{s \geq 0}$ follows from the fact that $s \mapsto C^{-1}W(s)x$, from $[r_i, r_j]$ into X, is continuous, hence uniformly continuous, for any $x \in Y, i, j \in \mathbf{N}$. Since Y is a Frechet space, the strong continuity implies that $\{T(t)\}_{t \geq 0}$ is locally equicontinuous (see [28]).

Let $\tilde{A}$ be the generator of $\{T(s)\}_{s \geq 0}$. If $x \in D(\tilde{A})$, then $\forall s > 0$,

$$\frac{1}{s}(W(s)x - Cx) = C\left[\frac{1}{s}(T(s)x - x)\right],$$

which converges to $C\tilde{A}x$, as $s \to 0$. Thus $x \in D(A)$, with $Ax = \tilde{A}x \in Y$.

This is saying that $\tilde{A} \subseteq A|_Y$. To see that $A|_Y \subseteq \tilde{A}$, suppose $x \in D(A|_Y)$. Then it is not hard to see that $C \in L(Y)$, thus, since

$$CT(t)x - Cx = W(t)x - Cx = \int_0^t W(s)Ax\,ds = C\left(\int_0^t T(s)Ax\,ds\right),$$

and C is injective, it follows that

$$T(t)x - x = \int_0^t T(s)Ax\,ds,$$

$\forall t \geq 0$, so that $x \in D(\tilde{A})$ and $\tilde{A} = A|_Y$, as desired.

It is clear that $Y \hookrightarrow X$, by choosing $r_i = 0, r_j > 0$. If $x = Cz \in Im(C)$, then $t \mapsto C^{-1}W(t)x$ is continuous and $\forall i, j \in \mathbf{N}$,

$$||x||_{i,j} \leq (\sup\{||W(t)|| | t \in [r_i, r_j]\}) \, ||C^{-1}x||,$$

thus $Im(C) \hookrightarrow Y$. ∎

Proof of Corollary 3.9: This follows from Theorem 3.8 exactly as Corollary 3.2 followed from Theorem 3.1. ∎

Proof of Propositions 3.11 and 3.14: Let $W(t) \equiv e^{tA|_Y}C$. Strong uniform continuity follows from the fact that $Im(C) \subseteq Y$ and $Y \hookrightarrow X$. It is not hard to show that, since $CA \subseteq AC$, C leaves Y invariant and commutes with $e^{tA|_Y}$. This implies that $W(t)W(s) = W(t+s)C, \forall s, t \geq 0$.

Thus $\{W(t)\}_{t\geq 0}$ is a C-semigroup. Let $\tilde{A}$ be the generator of $\{W(t)\}_{t\geq 0}$. Note that, $\forall y \in D(A)$, $W(t)y$ is the unique solution of (1.1), with $x = Cy$, since $C(D(A)) \subseteq D(A|_Y)$. Thus, $\frac{d}{dt}W(t)x|_{t=0}$ exists and equals $ACy = CAy$, so that $y \in D(\tilde{A})$ and $\tilde{A}y = Ay$, that is, $\tilde{A}$ is an extension of A. ∎

Proof of Propositions 3.12 and 3.15: For $x \in X$, let $W(t)x \equiv \tilde{C}e^{tB}x$. Boundedness and strong uniform continuity follows from the fact that $[\tilde{C}(W)] \hookrightarrow X$. Since $\tilde{C}e^{tB} = e^{tB}\tilde{C}$, $\{W(t)\}_{t\geq 0}$ is a C-semigroup.

Let $\tilde{A}$ be the generator of $\{W(t)\}_{t\geq 0}$ and suppose $x \in D(A)$. Since $x \in D(B)$ and $[\tilde{C}(W)] \hookrightarrow X$, $W(t)x$ is differentiable, with $\frac{d}{dt}W(t)x|_{t=0} = \tilde{C}Bx = CAx$. Thus $x \in D(\tilde{A})$, with $\tilde{A}x = Ax$, so that an extension of A generates $\{W(t)\}_{t\geq 0}$. Since $X \hookrightarrow W$, $\mathcal{D}_W \subseteq Im(C) \cap D(B|_X)$. Thus, by Proposition 2.4, $\tilde{A} = C^{-1}AC$. ∎

Proof of Theorem 3.16: This follows from Theorem 3.1 and Propositions 3.11, 3.12 and 2.4. ∎

Proof of Theorem 3.17: This follows from Theorem 3.8 and Propositions 3.14, 3.15 and 2.4. ∎

Proof of Corollary 3.18: By translating A if necessary, we may assume that all semigroups, integrated semigroups and C-semigroups are exponentially decaying, hence strongly uniformly continuous.

Corollary 3.18 now follows from Theorem 3.16 and the fact that A generates an exponentially bounded n-times integrated semigroup if and only if $\exists w > 0$ such that $(w, \infty) \subseteq \rho(A)$ and A generates an exponentially bounded $(r - A)^{-n}$-semigroup, $\forall r > w$ (see [15]). ∎

IV. Examples

We restrict ourselves here to examples simple enough so that $C^{-1}W(t)$ may be calculated, to illustrate the construction of the interpolating space Y. Other examples, where $\{W(t)\}_{t\geq 0}$ is constructed explicitly, but $C^{-1}W(t)$ might be less obvious, may be found in [15], [16], [17] and [18].

Example 4.1: Let $X \equiv C_0([0, \infty)), Af(x) \equiv xf(x)$, then A generates the C-semigroup

$$W(t)f(x) \equiv e^{-x^2} e^{tx} f(x),$$

where $C \equiv W(0)$. Then Y, from Theorem 3.7, is $\{f \in X \,|\, x \mapsto e^{tx} f(x) \in X, \forall t \geq 0\}$, with topology generated by the seminorms

$$\|f\|_{i,j} \equiv \sup\{|e^{tx} f(x)| \,|\, x \in \mathbf{R}, t \in [r_i, r_j]\},$$

and $e^{tA|_Y} f(x) = e^{tx} f(x), \forall f \in Y, t \geq 0$.

It is clear how to extend this construction to arbitrary multiplication operators, A. Our space Y is similar to a construction in [7], [8], where a Cauchy problem in population genetics is considered. ∎

For Examples 4.2 through 4.4, we will write $f(D)$, where

$$D \equiv i \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_k} \right),$$

for the operator

$$f(D)g \equiv (\mathcal{F}^{-1} f) * g,$$

with domain equal to the Schwartz space of rapidly decreasing functions, where $\mathcal{F}$ is the k dimensional Fourier transform, defined on the space of tempered distributions, f and g are tempered distributions and $*$ is convolution.

Example 4.2 : Let $A \equiv i\Delta$, on $L^p(\mathbf{R}^k)$ $(1 \le p \le \infty)$, the closure of $-i|D|^2$. For appropriate n, it is shown in [24] and [1] that A generates an n-times integrated semigroup; this implies(see [15]) that A generates a bounded, strongly uniformly continuous $(1-A)^{-n}$-semigroup, given by

$$W(t) \equiv (f_t h_n)(D), \quad \text{where } f_t(x) \equiv e^{-it|x|^2}, \quad h_n(x) \equiv (1 + i|x|^2)^{-n}.$$

Here Y, of Theorem 3.1, would be
$\{g \,|\, t \mapsto f_t(D)g$ is a bounded, strongly uniformly continuous map from $[0,\infty)$ into $L^p(\mathbf{R}^k)\}$, with $\|g\|_Y \equiv \sup\{\|f_t(D)g\| \,|\, t \ge 0\}$. ∎

Example 4.3: Let $A \equiv -\Delta$, on $L^p(\mathbf{R}^k)(1 \le p < \infty)$, the closure of $|D^2|$. The operator A generates an $e^{-\Delta^2}$-semigroup, given by

$$W(t) \equiv (g_t h)(D), \quad \text{where } g_t(x) \equiv e^{t|x|^2}, \quad h(x) \equiv e^{-|x|^4}.$$

This is shown in [16], for $k = 1$; the same construction may be used for arbitrary k. It would seem natural to construct Y, from Theorem 3.7, analogously to the previous example; the difficulty is that g_t is not a tempered distribution, thus $g_t(D)$ may not be defined.

For $p = 2$, one may use the fact that the Fourier transform is a unitary map, to show that

$$Y = \{f \in L^2(\mathbf{R}^k) |\, t \mapsto g_t \mathcal{F}^{-1} f \text{ is a continuous map from } [0,\infty) \text{ into } L^2(\mathbf{R}^k)\},$$

with topology generated by the seminorms

$$\|f\|_{i,j} \equiv \sup\{\|g_t \mathcal{F}^{-1} f\|_2 \,|\, t \in [r_i, r_j]\}.$$

For general p, if Ω is a bounded subset of $\mathbf{R}^k$ with smooth boundary, then $A \equiv -\Delta$, on $L^p(\Omega)$, with $D(A) \equiv W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$, also generates an e^{-A^2}-semigroup. Here we do not have the Fourier transform; the C-semigroup generated by A is defined with an unbounded analogue of the Cauchy integral formula, similarly to the construction of fractional powers, using only the fact that $-A$ generates a strongly continuous holomorphic semigroup (see [16]).

The operator in the previous paragraph gives us the backwards heat equation. The Cauchy problem for the Laplace equation may be similarly dealt with using C-semigroups, after the usual matrix reduction to a first order problem (see [16]).
∎

Example 4.4: Let X and A be as in Remark 3.6. Then Y equals

$$\{f \in C_0^1([0,\infty) \cap C^1(\mathbf{R}) | f' \text{ is bounded and uniformly continuous}\},$$

with $\|f\|_Y \equiv \sup_{x\in\mathbf{R}} |f(x)| + \sup_{x\in\mathbf{R}} |f'(x)|$.

More generally, if $X \equiv \bigcap_{k=0}^n C_0^k([k,\infty))$ with $\|f\| \equiv \sum_{k=0}^n \sup_{x\geq k} |f^{(k)}(x)|$ and $A \equiv -\frac{d}{dx}, D(A) \equiv \{f \in X | f' \in X\}$, then A generates a bounded strongly uniformly continuous $(1-A)^{-n}$-semigroup, and Y, from Theorem 3.1, equals $C_0^n([0,\infty))$, $\|f\|_Y = \sum_{k=0}^n \sup_{x\geq 0} |f^{(k)}(x)|$.

Similarly, let $X \equiv \{g \in \bigcap_{k=0}^\infty C_0^k([k,\infty)) \,|\, \|g\| < \infty\}$, where

$$\|g\| \equiv \sup\{|g^{(k)}(x)| \,|\, k \in \mathbf{N} \cup \{0\},\ x \in [k,\infty)\}.$$

Then it may be shown that A generates an $f(D)$-semigroup, whenever f is in the Schwartz space and $f(D)$ is injective; for example, $f(x) = e^{-x^2}$. In this case, Y would be $\{g \in \bigcap_{k=0}^\infty C_0^k([0,\infty)) \,|\, \|g\|_Y < \infty\}$, where

$$\|g\|_Y \equiv \sup\{|g^{(k)}(x)| \,|\, k \in \mathbf{N}\{0\},\ x \in [0,\infty)\}.$$

∎

Example 4.5: Suppose B is closed, $D(B)$ is dense and $(-\infty,0) \subseteq \rho(B)$, with $\{\|r(r+B)^{-1}\| \,|\, r > 0\}$ bounded. For $t > 0$, let B^t be the usual fractional power of B (see, for example, [3], [21]). Then $\{B^t e^{-B^{\frac{1}{2}}}\}_{t\geq 0}$ is an $e^{-B^{\frac{1}{2}}}$-semigroup (note that $-B^{\frac{1}{2}}$ is constructed in such a way that it generates a strongly continuous holomorphic semigroup $e^{-tB^{\frac{1}{2}}}$) (see [14]).

Let Y be as in Theorem 3.7. Then Y is an extension of the locally convex space $C^\infty(B)$, with seminorms $\|x\|_k \equiv \|B^k x\|$, in the sense that $C^\infty(B) \subseteq Y$, and the topology of Y, restricted to $C^\infty(B)$, is equivalent. ∎

V. Fractional powers of generators of bounded C-semigroups

In this section, we give an example of how section III may be used to translate results about strongly continuous semigroups into results about C-semigroups. Other places where fractional powers that do not generate strongly continuous semigroups are considered are [41], [43], [29], [30] and [31].

THEOREM 5.1: *Suppose $-A$ generates a strongly uniformly continuous bounded C-semigroup. Then $\exists$ a family of operators $\{A_r\}_{0\leq r\leq 1}$ such that $A_1 = A, A_0 = I$ and*

(1) *$-A_r$ generates a bounded holomorphic C-semigroup of angle $\frac{\pi}{2}(1-r)$, whenever $0 \leq r < 1$; and*

(2) *Suppose $0 \leq r, s \leq 1$ and $r + s \leq 1$. Then*

$$C(D(A)) \subseteq D(A_r A_s) \cap D(A_{r+s}),$$

with $A_r A_s x = A_{r+s} x, \forall x \in C(D(A))$;

(3) *$\lim_{r\to s} A_r x = A_s x, \forall x \in C(D(A))$, whenever $0 \leq r, s \leq 1$.*

If $\rho(A)$ is nonempty and $-A$ generates a uniformly continuous bounded holomorphic C-semigroup, then

(4) *$\overline{(A_{\frac{1}{k}})^k} = A, \forall k \in \mathbf{N}$.*

Remark 5.2: The property of generating a bounded integrated semigroup is not preserved by this construction. Consider the following example. Define A on $X \times X$ by

$$A \equiv \begin{bmatrix} G & G^2 \\ 0 & G \end{bmatrix}, \quad D(A) \equiv \{\vec{x} | G^{-1}x_1 + x_2 \in D(G^2), x_2 \in D(G)\},$$

where $-G$ generates an exponentially decaying strongly continuous semigroup. Then $-A$ generates a bounded twice integrated semigroup and a bounded G^{-2}-semigroup (see [15] and [39]). Following the construction of Theorem 5.1, it is not hard to see that, at least for $x \in D(A^3)$,

$$A_{\frac{1}{2}} x = \begin{bmatrix} G^{\frac{1}{2}} & \frac{1}{2} G^{\frac{3}{2}} \\ 0 & G^{\frac{1}{2}} \end{bmatrix} x.$$

It may be shown(see [15]) that, for any complex λ, $(\lambda - A_{\frac{1}{2}})^{-1}$ is unbounded. Thus $-A_{\frac{1}{2}}$ cannot generate an integrated semigroup, because $\rho(A_{\frac{1}{2}})$ is empty. ∎

Proof of Theorem 5.1: Let Y be as in Theorem 3.1. Let $(A|_Y)^r$ be the usual fractional power of $A|_Y$ (see, for example, [3] or [21]), so that $\{e^{-t(A|_Y)^r}\}_{t\geq 0}$ is a bounded strongly continuous holomorphic semigroup of angle $\frac{\pi}{2}(1-r)$. Let $-A_r$ be defined to be the generator of the bounded holomorphic C-semigroup

$\{e^{-t(A|_Y)^r}C\}_{t\geq 0}$. (2) and (3) follow from the fact that $C(D(A)) \subseteq D(A|_Y)$. Now suppose that $\rho(A)$ is nonempty. Arguments identical to those in [13] imply that an extension of $-(A_{\frac{1}{k}})^k, -\tilde{A}$, generates a bounded holomorphic C-semigroup, hence $(A_{\frac{1}{k}})^k$ is closable. (2) implies that $((A^{\frac{1}{k}}))^k x = Ax, \forall x \in C(D(A))$. Since $\rho(A)$ is nonempty, this is a core for A (see [13]), thus $A \subseteq \overline{(A_{\frac{1}{k}})^k} \subseteq \tilde{A}$. Since $\rho(A)$ is nonempty, and $-\tilde{A}$ generates a C-semigroup, this implies that $A = \tilde{A}$ (see Proposition 2.5), so that $A = \overline{(A_{\frac{1}{k}})^k}$, as desired. ∎

VI. Holomorphic interpolations and extrapolations

It is desirable to have the analyticity of a C-semigroup preserved by interpolations and extrapolations.

Definition 6.1: $S_\theta \equiv \{re^{i\phi} | r > 0, |\phi| < \theta\}$. ∎

Definition 6.2: (see [10], [19]). The C-semigroup $\{W(t)\}_{t\geq 0}$ is a **uniformly bounded holomorphic C-semigroup of angle θ** if it extends to a family of bounded operators $\{W(z)\}_{z\in\overline{S_\theta}}$ satisfying

(1) The map $z \mapsto W(z)$, from S_θ into $L(X)$, is holomorphic.

(2) $W(z)W(w) = CW(z+w), \forall z, w \in \overline{S_\theta}$.

(3) $W(z)$ is bounded and strongly continuous on $\overline{S_\theta}$.

We will call $\{W(z)\}$ a **strongly uniformly continuous uniformly bounded holomorphic C-semigroup of angle θ** if the continuity in (3) is uniform. As with strongly uniformly continuous bounded C-semigroups, it is sufficient, in Definition 6.2, to have the generator densely defined.

$\{W(t)\}_{t\geq 0}$ is a **bounded holomorphic C-semigroup of angle θ** if, $\forall \psi < \theta$, $\{W(t)\}_{t\geq 0}$ is a uniformly bounded holomorphic C-semigroup of angle ψ.

THEOREM 6.3: *Suppose A generates a strongly uniformly continuous uniformly bounded holomorphic C-semigroup of angle θ. Then $\exists$ a Banach space Y such that*

(1) *$A|_Y$ generates a uniformly bounded holomorphic strongly continuous semigroup of angle θ,*

(2) *$[C(X)] \hookrightarrow Y \hookrightarrow X$,*

(3) *$W(z) = e^{zA|_Y}C, \forall z \in S_\theta$.*

Y may be chosen so that $\|x\|_Y \equiv \sup\{\|C^{-1}W(z)x\| \,|\, z \in \overline{S_\theta}\}$.

COROLLARY 6.4: *Suppose A generates a strongly uniformly continuous uniformly bounded holomorphic C-semigroup, $\{W(z)\}_{z\in\overline{S_\theta}}$, of angle θ. Then $\exists$ a Banach space W and an operator B such that*

(1) *C extends to a bounded operator on W,*
(2) *B generates a uniformly bounded holomorphic strongly continuous semigroup of angle θ on W, such that $e^{zB}C = Ce^{zB}, \forall z \in \overline{S_\theta}$,*
(3) *$Ce^{zB}x = W(z)x, \forall z \in \overline{S_\theta}, x \in X$,*
(4) *$A = B|_X$ and*
(5) *$[C(W)] \hookrightarrow X \hookrightarrow W$.*

Open Question 6.5: It is clear from Theorem 6.3 that, when $D(A)$ is dense and A generates a bounded holomorphic C-semigroup of angle θ, then $\forall\psi < \theta, \exists$ an interpolation space Y_ψ, on which A generates a uniformly bounded holomorphic strongly continuous semigroup of angle ψ. It is not clear if there exists an interpolation (Banach) space on which A generates a bounded holomorphic strongly continuous semigroup of angle θ. ∎

Open Question 6.6: Suppose A generates a strongly uniformly continuous uniformly bounded holomorphic C-semigroup, Y_1 is the interpolation space of Theorem 6.3 and Y_2 is the interpolation space of Theorem 3.1. It is clear that $Y_1 \subseteq Y_2$. Are Y_1 and Y_2 equal? ∎

One way to answer this open question in the negative would be to construct a bounded strongly continuous semigroup $\{e^{tA}\}$ such that $\{e^{tA}C\}$ extends to a strongly uniformly continuous uniformly bounded holomorphic C-semigroup but $\{e^{tA}\}$ does not. It would also be sufficient to produce an operator, A, that generates a strongly uniformly continuous uniformly bounded holomorphic C-semigroup, for which (1.1) has a bounded solution that is not analytic.

An affirmative answer to this open question would also answer Open Question 6.5 in the affirmative, since it would then follow that $Y_{\psi_1} = Y_{\psi_2}, \forall\psi_1, \psi_2 < \theta$. ∎

Except for the following lemma, most of the proof of Theorem 6.3 is identical to the proof of Theorem 3.1.

LEMMA 6.7: *Suppose A generates a uniformly bounded holomorphic*

C-semigroup, $\{W(z)\}_{z\in\overline{S_\theta}}$ of angle θ. Then, $\forall y \in S_\theta$,

$$\sup_{w\in\overline{S_\theta}+y} ||\frac{1}{z}[W(w+z) - W(w)] - AW(w)||$$

converges to zero, as $|z|$ goes to zero.

One might call this being **uniformly analytic on subsectors.**

Proof: Let $M \equiv \sup_{z\in\overline{S_\theta}}\{||W(z)||\}$. For fixed z, let $\phi \equiv \arg(z)$. Then, for $|z|$ sufficiently small,

$$\begin{aligned}
&\sup_{w\in\overline{S_\theta}+y} ||\frac{1}{z}[W(w+z) - W(w)] - AW(w)|| \\
&\leq \sup_{w\in\overline{S_\theta}+y} ||\frac{1}{|z|}\int_0^{|z|} AW(se^{i\phi}+w) - AW(w)\,ds|| \\
&\leq \sup_{w\in\overline{S_\theta}+y}\ \sup_{0\leq s\leq |z|} ||AW(se^{i\phi}+w) - AW(w)|| \\
&= \sup_{w\in\overline{S_\theta}+y}\ \sup_{0\leq s\leq |z|} ||\int_0^s A^2W(re^{i\phi}+w)\,dr|| \\
&\leq \sup_{w\in\overline{S_\theta}+y}\ \sup_{|r|\leq |z|} |z|||A^2W(r+w)|| \\
&\leq \sup_{w\in\overline{S_\theta}+y}\ \sup_{|r|\leq |z|} |z|\frac{M}{|r+w|^2} \leq \frac{M|z|}{(|y|-|z|)^2}.
\end{aligned}$$

It is clear that this converges to zero, as $|z|$ converges to zero . The second to last inequality in the argument above follows from the Cauchy integral formula for derivatives of functions holomorphic in a sector, as with strongly continuous holomorphic semigroups. ∎

Proof of Theorem 6.3: Let Y be the set of all x such that $||x||_Y$ is finite and the map $w \mapsto C^{-1}W(w)x$ is a uniformly continuous map from $\overline{S_\theta}$ into X, and a uniformly analytic map from subsectors of S_θ into X, that is, for all $y \in S_\theta$,

$$\sup_{w\in\overline{S_\theta}+y} ||\frac{1}{z}\left[C^{-1}W(w+z)x - C^{-1}W(w)x\right] - C^{-1}W'(w)x||$$

converges to zero, as $|z|$ goes to zero.

Then the same arguments as in the proof of Theorem 3.1 imply that Y is a Banach space and $A|_Y$ generates a uniformly bounded strongly continuous holomorphic semigroup of angle θ, $T(z) \equiv C^{-1}W(z)$. As in the proof of Theorem 3.1, Lemma 6.7 implies that $[C(X)] \hookrightarrow Y$. ∎

VII. Perturbations of generators of C-semigroups

We will write $(A+B)$ to mean the operator with domain $D(A) \cap D(B)$. The natural setting for perturbations is a noncommuting version of C-semigroups, introduced in [17].

Definition 7.1: The strongly continuous family of bounded operators $\{W(t)\}_{t\geq 0}$ is an **exponentially bounded mild C-existence family for A** if A is closable and $\exists w > 0$ such that

(1) $||W(t)||$ is $O(e^{wt})$,

(2) $(r - \overline{A})$ is injective, $\forall r > w$,

(3) $Im(C) \subseteq Im(r - A), \forall r > w$,

(4) The map $t \mapsto \int_0^t W(s)x\,ds \in C\left([0,\infty), [D(A)]\right), \forall x \in X$,

(5) $||A\left(\int_0^t W(s)x\,ds\right)||$ is $O(e^{wt}), \forall x \in X$, and

$$(r-A)^{-1}Cx = \int_0^\infty e^{-rt}W(t)x\,dt, \quad \forall x \in X, \quad r > w.$$

∎

The following may be found in [17].

PROPOSITION 7.2: *Suppose $\exists$ an exponentially bounded mild C-existence family for a restriction of A. Then (1.1) has a unique exponentially bounded mild solution, $\forall x \in Im(C)$.*

THEOREM 7.3: *Suppose*

(1) *A generates a bounded uniformly continuous C_1-semigroup $\{W(t)\}_{t\geq 0}$ that commutes with C_2,*

(2) *$C_1^{-1}C_2 \in L(X)$,*

(3) *B is closed in X and*

(4) *$Z \equiv \{x | t \mapsto C_2^{-1}W(t)x$ is a bounded uniformly continuous map from $[0,\infty)$ into $X\} \subseteq D(B)$, with $t \mapsto C_2^{-1}W(t)Bx$ bounded and uniformly continuous, $\forall x \in Z$.*

Then $\exists$ a bounded mild C_2-existence family for a restriction of $(A+B)$.

What is of interest here is that B could be unbounded, even relative to A. Consider the following example.

Example 7.4: Suppose G generates a bounded strongly continuous group, $\{T(t)\}_{t\geq 0}$, on X, B is closed in X, $r \in \rho(G)$ and $D(G^n) \subseteq D(B)$. Define A, on $X \times X$, by

$$A \equiv \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}, \quad D(A) \equiv D(G) \times D(G).$$

Let

$$C_2 \equiv \begin{bmatrix} I & 0 \\ 0 & (r-G)^{-n} \end{bmatrix}.$$

Then Theorem 7.3 may be used to show that $\exists$ a bounded mild C_2-existence family for

$$\begin{bmatrix} G & B \\ 0 & G \end{bmatrix},$$

since

$$\left|\left|C_2^{-1} \begin{bmatrix} T(t) & 0 \\ 0 & T(t) \end{bmatrix} \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} x\right|\right| = ||T(t)[B(r-G)^{-n}](r-G)^n x_2||,$$

whenever $x_2 \in D(G^n)$. ∎

As a corollary of Theorem 7.3, we get the best perturbation result for C-semigroups currently known (see [17], [47] or [20]).

COROLLARY 7.5: *Suppose $B \in L(X, [C(X)])$ and A generates a bounded uniformly continuous C-semigroup. Then $\exists$ a bounded mild C-existence family for a restriction of $(A+B)$.*

Proof of Theorem 7.3: Let Y be as in Theorem 3.1. Then $Z = [C_1^{-1}C_2(Y)]$. Since C_2 commutes with $W(t)$, it commutes with $e^{tA|_Y}$, thus $A|_Z$ also generates a strongly continuous contraction semigroup. Since $Z \hookrightarrow X$, B is closed in Z. Thus, by (4) and the Closed Graph Theorem, $B|_Z \in L(Z)$. This implies that $(A|_Z + B|_Z)$ generates a strongly continuous contraction semigroup $\{T(t)\}_{t\geq 0}$, on Z. It is well-known that $(0,\infty) \subseteq \rho(A|_Z + B|_Z)$, with

$$(s-(A|_Z+B|_Z))^{-1}x = \int_0^\infty e^{-st}T(t)x\,dt,$$

$\forall s > 0, x \in Z$. Since $[C_2(X)] \hookrightarrow Z$, this implies that, if $W(t) \equiv T(t)C_2$, then

$$(s - (A|_Z + B|_Z))^{-1} C_2 x = \int_0^\infty e^{-st} W(t)x\, dt,$$

$\forall s > 0$, $x \in X$. Since $\int_0^t W(s)x\, ds = \int_0^t T(s)C_2 x\, ds \in D(A|_Z)$, with

$$(A|_Z + B|_Z)\left(\int_0^t T(s)C_2 x\, ds\right) = T(t)C_2 x - C_2 x,$$

$W(t)$ satisfies (4) and (5) of Definition 7.1. Thus $\{W(t)\}_{t \geq 0}$ is a bounded mild C_2-existence family for $(A|_Z + B|_Z)$, a restriction of $(A + B)$. ∎

Proof of Corollary 7.5: $C^{-1}B \in L(X)$, thus $W(t)Bx \in Im(C), \forall t \geq 0, x \in X$, with $C^{-1}W(t)Bx = W(t)C^{-1}Bx, \forall t \geq 0$, so that this follows from Theorem 7.3, with $C_1 = C_2 = C$. ∎

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